IR divergences and Regge limits of subleading-color contributions to the four-gluon amplitude in $\mathcal{N} = 4$ SYM theory

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Abstract: We derive a compact all-loop-order expression for the IR-divergent part of the $\mathcal{N} = 4$ SYM four-gluon amplitude, which includes both planar and all subleading-color contributions, based on the assumption that the higher-loop soft anomalous dimension matrices are proportional to the one-loop soft anomalous dimension matrix, as has been recently conjectured.

We also consider the Regge limit of the four-gluon amplitude, and we present evidence that the leading logarithmic growth of the subleading-color amplitudes is less severe than that of the planar amplitudes. We examine possible $1/N^2$ corrections to the gluon Regge trajectory, previously obtained in the planar limit from the BDS ansatz. The double-trace amplitudes have Regge behavior as well, with a nonsense-choosing Regge trajectory and a Regge cut which first emerges at three loops.

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Contents

1 Introduction

Over the past decade, there has been much interest in $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills (SYM) theory, in part because of its relation to string theory via the AdS/CFT correspondence, and because of the possibility that, in the large N (planar) limit, the theory may be integrable and solvable.

Recent progress on the perturbative structure of the theory has been motivated by the discovery of an iterative structure of the loop amplitudes [1] which together with an analysis of IR divergences [2–5] led to the fruitful BDS conjecture [6] for the all-loop-orders MHV planar n-gluon amplitude. This conjecture has been shown to be a consequence of

dual conformal invariance[1] for $n = 4$ and 5, but for $n \geq 6$ must be modified [12–16], though the exact form of the correction is not yet known. In refs. [8, 17, 18] the BDS ansatz for the planar four-gluon amplitude was shown to imply exact Regge behavior, and the gluon Regge trajectory (in the planar limit) was computed. The Regge behavior of higher-point planar amplitudes has been explored in refs. [19–22].

While the leading-color (planar) amplitudes have been under intense investigation, sub-leading-color amplitudes have received much less scrutiny. Two-loop subleading-color four-gluon amplitudes [23] can be written explicitly [24, 25] through $\mathcal{O}(\epsilon^0)$ in a Laurent expansion in the dimensional regulator $\epsilon = (4-D)/2$, and three-loop subleading-color four-gluon amplitudes are known in terms of a basis of scalar integrals [26], but no BDS-type ansatz is known for general L-loop subleading-color amplitudes. In previous work [27], we derived explicit expressions for the IR-divergent part of subleading-color four-gluon amplitudes through three loops, and made several conjectures about the extension of these expressions to arbitrary loop order. In the first part of this paper, we derive (using an assumption explicitly stated below) an all-loop-orders expression for the IR-divergent part of the four-gluon amplitude, confirming and extending the conjectures made in ref. [27].

The BDS ansatz was guided by an analysis of the IR divergences of loop amplitudes [2–5]. In the planar limit, the IR divergences depend on two functions of the coupling: the soft (cusp) anomalous dimension $\gamma(a)$ and the collinear anomalous dimension $\mathcal{G}_0(a)$. The IR divergences of subleading-color amplitudes depend not only on $\gamma(a)$ and $\mathcal{G}_0(a)$ but also on a soft anomalous dimension matrix $\mathbf{\Gamma}(a)$. It was shown [28, 29] that the two-loop soft anomalous dimension matrix is proportional to the one-loop matrix

$$\mathbf{\Gamma}^{(2)} = \frac{\gamma^{(2)}}{\gamma^{(1)}} \mathbf{\Gamma}^{(1)} \tag{1.1}$$

where $\mathbf{\Gamma}(a) = \sum_{\ell=1}^{\infty} a^\ell \mathbf{\Gamma}^{(\ell)}$ and $\gamma(a) = \sum_{\ell=1}^{\infty} a^\ell \gamma^{(\ell)}$. Dixon recently established the analogous proportionality for the matter-dependent part of the three-loop soft anomalous dimension matrix [30]. An all-orders form for $\mathbf{\Gamma}(a)$ has been conjectured [26, 31–33], which in the case of $\mathcal{N} = 4$ SYM theory reduces to

$$\mathbf{\Gamma}^{(\ell)} = \frac{\gamma^{(\ell)}}{\gamma^{(1)}} \mathbf{\Gamma}^{(1)} \tag{1.2}$$

generalizing eq. (1.1). In this paper, we will assume that eq. (1.2) holds for all ℓ, and thus that the $\mathbf{\Gamma}^{(\ell)}$ are mutually commuting, to derive a compact formula for the all-loop-order IR divergences of the $\mathcal{N} = 4$ SYM four-gluon amplitude

$$|A(\epsilon)\rangle = \exp\left[\sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{G}^{(\ell)}(\ell\epsilon) \right] |H(\epsilon)\rangle \tag{1.3}$$

where $|H(\epsilon)\rangle$ denotes the short-distance, IR-finite, contribution to the amplitude and

$$\mathbf{G}^{(\ell)}(\epsilon) = \frac{N^\ell}{2} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[-\left(\frac{\gamma^{(\ell)}}{\epsilon^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\epsilon} \right) \mathbb{1} + \frac{\gamma^{(\ell)}}{4\epsilon} \mathbf{\Gamma}^{(1)} \right] \tag{1.4}$$

---

[1]More precisely, anomalous dual conformal symmetry uniquely fixes the form of light-like Wilson loops for $n = 4$ and $n = 5$ [7–9], and much evidence has accumulated for the equivalence of Wilson loops to MHV planar amplitudes [8–14].

where $\mathcal{G}_0(a) = \sum_{\ell=1}^{\infty} a^\ell \mathcal{G}_0^{(\ell)}$, and μ and Q are the renormalization and factorization scales respectively. We use this to derive expressions for specific subleading-color amplitudes, and to confirm and extend some of the conjectures made in ref. [27].

In the second part of this paper, in an effort to see whether the iterative structures that play such an important role in MHV planar amplitudes might also be present in subleading-color amplitudes, we consider the Regge limit ($s \to \infty$ with t fixed) of the four-gluon amplitude to all orders in perturbation theory. We present evidence that the planar L-loop amplitude has $\log^L(-s/t)$ leading log behavior, while subleading-color amplitudes only go as $\log^{L-1}(-s/t)$, using the IR-divergent contributions as a guide. (To fully prove this behavior would require knowing the IR-finite parts of the amplitudes as well.)

The IR-divergent parts of the subleading-color amplitudes possess sufficient iterative structure to enable us (partially) to sum them (neglecting terms of $\mathcal{O}(t/s)$) to all orders in perturbation theory. There are no subleading-color corrections to the gluon Regge trajectory function through two loops, in agreement with the maximum transcendentality contribution [34] of the QCD gluon Regge trajectory [35, 36], although the Regge residue picks up a two-loop $1/N^2$ correction. We find Regge-type behavior for the IR-divergent terms of the (subleading-color) double-trace amplitudes. It remains to be seen whether these iterative structures extend beyond the Regge limit.

In section 2, we derive a compact all-loop-orders expression for the IR-divergent part of the $\mathcal{N} = 4$ SYM four-gluon amplitude, and in section 3 we use this to derive explicit expressions for subleading-color amplitudes. Section 4 examines the leading logarithmic behavior of leading- and subleading-color L-loop amplitudes in the Regge limit. In section 5, the leading logarithms are summed to obtain Regge trajectories. Conclusions are presented in section 6, and technical details are to be found in two appendices.

2 $\mathcal{N} = 4$ SYM IR divergences to all loops

In this section, we derive a compact all-loop-orders expression for the IR-divergent part of the $\mathcal{N} = 4$ SYM four-gluon amplitude in terms of anomalous dimensions $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$, soft anomalous dimension matrices $\mathbf{\Gamma}^{(\ell)}$, and the IR-finite parts of lower-loop amplitudes. This result relies on the assumption that the soft anomalous dimension matrices are mutually commuting, which follows if they are all proportional to $\mathbf{\Gamma}^{(1)}$, as has been recently conjectured [26, 31–33]. We then show that our expression is consistent with previous results at one, two, and three loops [4, 5, 27].

First, we decompose the four-gluon amplitude into a basis of traces of color generators

$$\mathcal{A}_{4-\text{gluon}}(1,2,3,4) = g^2 \sum_{i=1}^{9} A_{[i]} \, \mathcal{C}_{[i]} \tag{2.1}$$

where the color-ordered amplitudes $A_{[i]}$ depend on the momenta k_i and helicities of the

gluons, and we adopt the explicit basis of single and double traces [37]

$$
\begin{aligned}
\mathcal{C}_{[1]} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}), & \mathcal{C}_{[4]} &= \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}), & \mathcal{C}_{[7]} &= \mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4}) \\
\mathcal{C}_{[2]} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}), & \mathcal{C}_{[5]} &= \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}), & \mathcal{C}_{[8]} &= \mathrm{Tr}(T^{a_1}T^{a_3})\mathrm{Tr}(T^{a_2}T^{a_4}) \\
\mathcal{C}_{[3]} &= \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}), & \mathcal{C}_{[6]} &= \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}), & \mathcal{C}_{[9]} &= \mathrm{Tr}(T^{a_1}T^{a_4})\mathrm{Tr}(T^{a_2}T^{a_3}).
\end{aligned}
\tag{2.2}
$$

Here T^a are $\mathrm{SU}(N)$ generators in the fundamental representation, normalized according to $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. It is convenient to organize the color-ordered amplitudes $A_{[i]}$ into a vector in color space [3, 4]

$$
|A\rangle = \left(A_{[1]}, \, A_{[2]}, \, A_{[3]}, \, A_{[4]}, \, A_{[5]}, \, A_{[6]}, \, A_{[7]}, \, A_{[8]}, \, A_{[9]} \right)^T
\tag{2.3}
$$

where $(\cdots)^T$ denotes the transposed vector.

Next, we write the color-ordered amplitudes in a loop expansion

$$
|A\rangle = \sum_{L=0}^{\infty} a^L |A^{(L)}\rangle
\tag{2.4}
$$

where the natural 't Hooft loop expansion parameter is [6]

$$
a \equiv \frac{g^2 N}{8\pi^2} \left(4\pi \mathrm{e}^{-\gamma} \right)^{\epsilon} .
\tag{2.5}
$$

Here γ is Euler's constant, and the loop amplitudes are evaluated using dimensional regularization in $D = 4 - 2\epsilon$ dimensions. Although $\mathcal{N} = 4$ SYM theory is UV finite, the dimensionally-regularized amplitudes contain poles in ϵ due to IR divergences. We follow the approach of refs. [5, 29] to organize the IR divergences as

$$
\left| A\left(\frac{s_{ij}}{\mu^2}, a, \epsilon \right) \right\rangle = J\left(\frac{Q^2}{\mu^2}, a, \epsilon \right) \mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon \right) \left| H\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon \right) \right\rangle
\tag{2.6}
$$

where the prefactors J and $\mathbf{S}$ characterize the long-distance IR-divergent behavior, and $|H\rangle$, which is finite as $\epsilon \to 0$, characterizes the short-distance behavior of the amplitude. Also $s_{ij} = (k_i + k_j)^2$, μ is a renormalization scale, and Q is an arbitrary factorization scale which serves to separate the long- and short-distance behavior. Although Q was set equal to μ in ref. [29] for simplicity, we will keep it arbitrary. When we consider the Regge limit of the four-gluon amplitudes in sections 4 and 5, we will set Q^2 equal to the fixed momentum scale $-t$.

Because $\mathcal{N} = 4$ SYM theory is conformally invariant, the product of jet functions J may be explicitly evaluated as [6]

$$
J\left(\frac{Q^2}{\mu^2}, a, \epsilon \right) = \exp\left[-\frac{1}{2} \sum_{\ell=1}^{\infty} a^\ell \left(\frac{\mu^2}{Q^2} \right)^{\ell\epsilon} \left(\frac{\gamma^{(\ell)}}{(\ell\epsilon)^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\ell\epsilon} \right) \right]
\tag{2.7}
$$

where $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$ are the coefficients of the soft (or Wilson line cusp) and collinear anomalous dimensions of the gluon respectively

$$\gamma(a) = \sum_{\ell=1}^{\infty} a^\ell \gamma^{(\ell)} = 4a - 4\zeta_2 a^2 + 22\zeta_4 a^3 + \cdots$$

$$\mathcal{G}_0(a) = \sum_{\ell=1}^{\infty} a^\ell \mathcal{G}_0^{(\ell)} = -\zeta_3 a^2 + \left(4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3\right) a^3 + \cdots \tag{2.8}$$

The soft function $\mathbf{S}$, written in boldface to indicate that it is a matrix acting on the vector $|H\rangle$, is given by [5, 29]

$$\mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) = \mathrm{P}\exp\left[-\frac{1}{2}\int_0^{Q^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\mathbf{\Gamma}\left(\frac{s_{ij}}{Q^2}, \bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a, \epsilon\right)\right)\right] \tag{2.9}$$

where[2]

$$\mathbf{\Gamma}\left(\frac{s_{ij}}{Q^2}, a\right) = \sum_{\ell=1}^{\infty} a^\ell \mathbf{\Gamma}^{(\ell)}, \qquad \bar{a}\left(\frac{\mu^2}{\tilde{\mu}^2}, a, \epsilon\right) = \left(\frac{\mu^2}{\tilde{\mu}^2}\right)^\epsilon a. \tag{2.10}$$

The integral (2.9) is path-ordered, but this becomes irrelevant if the soft anomalous dimension matrices $\mathbf{\Gamma}^{(\ell)}$ all commute with one another. In ref. [28, 29] it was shown that $\mathbf{\Gamma}^{(2)} = \frac{1}{4}\gamma^{(2)}\mathbf{\Gamma}^{(1)}$, and in ref. [30] that $\mathbf{\Gamma}^{(3)} = \frac{1}{4}\gamma^{(3)}\mathbf{\Gamma}^{(1)}$ for the non pure gluon contributions. If we assume that

$$\mathbf{\Gamma}^{(\ell)} = \frac{\gamma^{(\ell)}}{4}\mathbf{\Gamma}^{(1)} \qquad (assumption) \tag{2.11}$$

holds for all ℓ in $\mathcal{N} = 4$ SYM theory,[3] as has been conjectured in refs. [26, 31–33], then the $\mathbf{\Gamma}^{(\ell)}$ indeed commute,[4] and we can explicitly integrate eq. (2.9) to obtain

$$\mathbf{S}\left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a, \epsilon\right) = \exp\left[\frac{1}{2}\sum_{\ell=1}^{\infty} a^\ell \left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon} \frac{\mathbf{\Gamma}^{(\ell)}}{\ell\epsilon}\right]. \tag{2.12}$$

Combining the exponents of the jet and soft functions into[5] [5, 27]

$$\mathbf{G}^{(\ell)}(\epsilon) = \frac{N^\ell}{2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \left[-\left(\frac{\gamma^{(\ell)}}{\epsilon^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\epsilon}\right)\mathbb{1} + \frac{1}{\epsilon}\mathbf{\Gamma}^{(\ell)}\right] \tag{2.13}$$

we may express the four-gluon amplitude in the compact form[6]

$$|A(\epsilon)\rangle = \exp\left[\sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell}\mathbf{G}^{(\ell)}(\ell\epsilon)\right]|H(\epsilon)\rangle \tag{2.14}$$

---

[2]We suppress the explicit dependence of $\mathbf{\Gamma}^{(\ell)}$ on s_{ij}/Q^2 to lighten the notation.

[3] Difficulties may arise at four loops, however, due to the possibility of quartic Casimir terms [30, 33, 38–40].

[4]The assumption that $\mathbf{\Gamma}^{(\ell)}$ commute was also used to simplify the IR divergences of QCD in ref. [32].

[5]In ref. [27], Q was set equal to μ.

[6]Henceforth we suppress s_{ij}, Q, μ, and a in the arguments of the amplitudes.

which will be very useful in extracting the IR-divergent parts of subleading-color amplitudes in section 3. The expression (2.14) is valid up to the number of loops L for which the set of soft anomalous dimension matrices $\{\mathbf{\Gamma}^{(\ell)} \mid \ell \leq L\}$ mutually commute, at least $L = 2$ and possibly to all orders.

We now briefly show that eq. (2.14) is consistent with previous results at one, two, and three loops [4, 5, 27]. Equations (3.13-3.15) of ref. [27] and their generalization to all L are compactly written as

$$|\tilde{A}^{(f)}(\epsilon)\rangle = \sum_{L=0}^{\infty} a^L |\tilde{A}^{(Lf)}(\epsilon)\rangle = \left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{F}^{(\ell)}(\epsilon) \right) |A(\epsilon)\rangle. \tag{2.15}$$

The $\mathbf{F}^{(\ell)}$ are chosen so as to cancel all the IR divergences in $|A(\epsilon)\rangle$, leaving an IR-finite expression $|\tilde{A}^{(f)}(\epsilon)\rangle$. In view of eq. (2.14), this can be accomplished by requiring

$$\left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{F}^{(\ell)}(\epsilon) \right) \exp\left[\sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{G}^{(\ell)}(\ell\epsilon) \right] = \mathbb{1}. \tag{2.16}$$

The $\mathbf{F}^{(\ell)}$ defined by eq. (2.16) may be written more explicitly as follows. In ref. [6], the functional $X[M]$ was defined via

$$1 + \sum_{\ell=1}^{\infty} a^\ell M^{(\ell)} \equiv \exp\left[\sum_{\ell=1}^{\infty} a^\ell \left(M^{(\ell)} - X^{(\ell)}[M] \right) \right] \tag{2.17}$$

thus, e.g., $X^{(1)}[M] = 0$, $X^{(2)}[M] = \frac{1}{2} \left[M^{(1)} \right]^2$, $X^{(3)}[M] = -\frac{1}{3} \left[M^{(1)} \right]^3 + M^{(1)} M^{(2)}$, etc. This functional was defined for scalar functions $M^{(\ell)}$, but we can also use it for commuting matrices. We have assumed that $\mathbf{\Gamma}^{(\ell)}$ and therefore $\mathbf{G}^{(\ell)}$ all commute with one another, and thus $\mathbf{F}^{(\ell)}$ do so as well as a consequence of eq. (2.16). Thus we can write

$$\left(\mathbb{1} - \sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{F}^{(\ell)}(\epsilon) \right) = \exp\left[\sum_{L=0}^{\infty} \frac{a^\ell}{N^\ell} \left(-\mathbf{F}^{(\ell)}(\epsilon) - X^{(\ell)}[-\mathbf{F}] \right) \right] \tag{2.18}$$

and so eq. (2.16) is equivalent to

$$\mathbf{F}^{(\ell)}(\epsilon) = -X^{(\ell)}[-\mathbf{F}] + \mathbf{G}^{(\ell)}(\ell\epsilon) \tag{2.19}$$

which defines $\mathbf{F}^{(\ell)}$ recursively in terms of $\mathbf{G}^{(\ell)}$ and $\mathbf{F}^{(\ell')}$ with $\ell' < \ell$. Equation (2.19) precisely agrees, in the case where the $\mathbf{F}^{(\ell)}$ commute with one another, with eqs. (3.16-3.18) of ref. [27][7] for $\ell \leq 3$, and provides their all-orders generalization. Equations (2.14)–(2.16) then imply

$$|\tilde{A}^{(f)}(\epsilon)\rangle = |H(\epsilon)\rangle \tag{2.20}$$

that is, the IR-finite function defined via eq. (2.15) is identical to the short-distance function defined in eq. (2.6).

In appendix A, we show how eq. (2.14) may also be used to easily obtain the IR-divergent part of the L-loop generalization [6] of the ABDK equation [1].

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[7] Based on the results of ref. [5].

3 IR divergences in the $1/N$ expansion

The L-loop color-ordered amplitudes may be written in a $1/N$ expansion as

$$|A^{(L)}(\epsilon)\rangle = \sum_{k=0}^{L} \frac{1}{N^k}|A^{(L,k)}(\epsilon)\rangle \tag{3.1}$$

where $|A^{(L,0)}\rangle$ are the leading-color (planar) amplitudes and $|A^{(L,k)}\rangle$, $1 \le k \le L$, are the subleading-color amplitudes. The L-loop planar amplitudes are predicted by the BDS ansatz [6], but no general expression is known for the L-loop subleading-color amplitudes (although exact expressions in terms of scalar integrals are known through three loops [26]). In this section, we will use the result (2.14) derived in section 2 to extract explicit expressions for the IR-divergent parts of subleading-color amplitudes. These will be useful in discussing the Regge limits of these amplitudes in sections 4 and 5.

We begin by expanding eq. (2.14):

$$|A(\epsilon)\rangle = \sum_{L=0}^{\infty}\sum_{k=0}^{L} \frac{a^L}{N^k}|A^{(L,k)}(\epsilon)\rangle = \prod_{\ell=1}^{\infty}\sum_{\{n_\ell\}} \frac{1}{n_\ell!}\left(a^\ell \frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell}\right)^{n_\ell} \sum_{\ell_0=0}^{\infty}\sum_{k_0=0}^{\ell_0} \frac{a^{\ell_0}}{N^{k_0}}|H^{(\ell_0,k_0)}(\epsilon)\rangle. \tag{3.2}$$

Assuming that the proportionality (2.11) holds, we use eq. (2.13) to write

$$\frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell} = \frac{1}{2}\left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon}\left[-\left(\frac{\gamma^{(\ell)}}{(\ell\epsilon)^2} + \frac{2\mathcal{G}_0^{(\ell)}}{\ell\epsilon}\right)\mathbb{1} + \frac{\gamma^{(\ell)}}{4\ell\epsilon}\mathbf{\Gamma}^{(1)}\right]. \tag{3.3}$$

The one-loop soft anomalous dimension matrix, defined by eq. (B.2), takes the form

$$\mathbf{\Gamma}^{(1)} = 2\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} + \frac{2}{N}\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \tag{3.4}$$

where explicit expressions for the momentum-dependent matrices α, β, γ, and δ are given in appendix B. Due to the assumption (2.11), the $1/N$ expansion of $\mathbf{G}^{(\ell)}(\ell\epsilon)/N^\ell$ has only two terms

$$\frac{\mathbf{G}^{(\ell)}(\ell\epsilon)}{N^\ell} = g_\ell + \frac{1}{N}f_\ell \tag{3.5}$$

where g_ℓ and f_ℓ can be read from eqs. (3.3) and (3.4). We rewrite eq. (3.2) as

$$|A(\epsilon)\rangle = \sum_{L=0}^{\infty}\sum_{k=0}^{L} \frac{a^L}{N^k}|A^{(L,k)}(\epsilon)\rangle = \prod_{\ell=1}^{\infty}\sum_{\{n_\ell\}} \frac{1}{n_\ell!}\left(a^\ell g_\ell + \frac{a^\ell}{N}f_\ell\right)^{n_\ell} \sum_{\ell_0=0}^{\infty}\sum_{k_0=0}^{\ell_0} \frac{a^{\ell_0}}{N^{k_0}}|H^{(\ell_0,k_0)}(\epsilon)\rangle \tag{3.6}$$

so that all N dependence is explicit.

Now consider an individual term on the r.h.s. of eq. (3.6). By counting powers of a and $1/N$, one sees that this term contributes to $|A^{(L,k)}(\epsilon)\rangle$, with

$$L = \ell_0 + \sum_{\ell=1}^{\infty}\ell n_\ell, \qquad k = k_0 + k_1 \tag{3.7}$$

where k_1 is the number of factors f_ℓ present in the term. From eqs. (3.3) and (3.4), it is apparent that g_ℓ has a double pole in ϵ, but f_ℓ only has a single pole. The leading IR pole in the term under consideration is therefore $1/\epsilon^p$, where

$$p = 2 \sum_{\ell=1}^{\infty} n_\ell - k_1 \,. \tag{3.8}$$

Combining eqs. (3.7) and (3.8), we find

$$p = 2L - k - \left[2 \sum_{\ell=1}^{\infty} (\ell-1)n_\ell + 2\ell_0 - k_0 \right] \,. \tag{3.9}$$

Since $k_0 \leq \ell_0$, the term in square brackets is non-negative, so the leading IR pole of $|A^{(L,k)}(\epsilon)\rangle$ is

$$|A^{(L,k)}(\epsilon)\rangle \sim \mathcal{O}\left(\frac{1}{\epsilon^{2L-k}} \right) \,. \tag{3.10}$$

This behavior was previously established in ref. [27] for amplitudes through $L = 3$.

3.1 Leading IR divergence of $A^{(L,k)}$

We now derive the coefficient of the leading IR pole of $|A^{(L,k)}(\epsilon)\rangle$. Terms in eq. (3.6) contribute to the leading IR pole only when the expression in square brackets in eq. (3.9) vanishes, which occurs when $n_\ell = 0$ for $\ell \geq 2$, and $\ell_0 = k_0 = 0$ (with n_1 unconstrained). In other words, the leading IR divergences are given by

$$|A(\epsilon)\rangle \sim \exp\left[a \frac{\mathbf{G}^{(1)}(\epsilon)}{N} \right] |A^{(0)}\rangle \qquad \text{(leading IR divergence)} \tag{3.11}$$

where $|H^{(0,0)}\rangle = |A^{(0)}\rangle$. This confirms a conjecture[8] made in ref. [27]. Recalling that

$$\frac{\mathbf{G}^{(1)}(\epsilon)}{N} = \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[-\frac{2}{\epsilon^2} \mathbb{1} + \frac{1}{\epsilon} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} + \frac{1}{N\epsilon} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right] \tag{3.12}$$

we use eq. (3.11) to obtain the coefficient of the leading IR pole

$$|A^{(L,k)}(\epsilon)\rangle = \frac{(-2)^{L-k}}{k!(L-k)!} \frac{1}{\epsilon^{2L-k}} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^k |A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-k-1}} \right) \,. \tag{3.13}$$

The leading IR pole of the planar amplitude is simply

$$|A^{(L,0)}(\epsilon)\rangle = \frac{(-2)^L}{L!\,\epsilon^{2L}} |A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-1}} \right) \tag{3.14}$$

---

[8] In that paper we expressed this in terms of $\mathbf{I}^{(1)}$, the operator introduced in ref. [3, 4], but as we showed there $\mathbf{I}^{(1)}$ and $\mathbf{G}^{(1)}$ only differ by terms subleading in ϵ.

with the rest of the IR divergences given by the (generalized) ABDK equation (see appendix A). The leading IR poles of the subleading-color amplitudes may be written explicitly using eqs. (B.4)–(B.7),

$$
|A^{(L,2m+1)}(\epsilon)\rangle \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.15)
$$

$$
= \left(\frac{-4iK}{stu}\right) \frac{(-1)^{L-1}2^{L-m}\left(X^2+Y^2+Z^2\right)^m(sY-tX)}{(2m+1)!(L-2m-1)!\,\epsilon^{2L-2m-1}}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}
+ \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-2}}\right)
$$

and

$$
|A^{(L,2m+2)}(\epsilon)\rangle \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.16)
$$

$$
= \left(\frac{-4iK}{stu}\right) \frac{(-1)^{L}2^{L-m-1}\left(X^2+Y^2+Z^2\right)^m(sY-tX)}{(2m+2)!(L-2m-2)!\epsilon^{2L-2m-2}}
\begin{pmatrix} X-Y \\ Z-X \\ Y-Z \\ Y-Z \\ Z-X \\ X-Y \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-3}}\right)
$$

where s, t, and u are the Mandelstam invariants, K depends on the momenta and helicity of the gluons, and is totally symmetric under permutations of the external legs, and

$$
X = \log\left(\frac{t}{u}\right), \qquad Y = \log\left(\frac{u}{s}\right), \qquad Z = \log\left(\frac{s}{t}\right). \qquad (3.17)
$$

The results (3.15) and (3.16) are generalizations of the expressions derived in ref. [27].

3.2 IR divergences of $A^{(L,L)}$

In the previous section, we derived the coefficient of the leading IR pole of the leading- and subleading-color amplitudes $|A^{(L,k)}\rangle$. It is also possible to use eq. (3.6) to derive further terms in the Laurent expansion.

In this section, we derive an expression for the IR divergences of the most subleading-color amplitude $|A^{(L,L)}\rangle$. The only terms in eq. (3.6) that contribute to $|A^{(L,L)}\rangle$ are those with as many factors of $1/N$ as of a. Thus, only f_1 and $|H^{(\ell_0,\ell_0)}\rangle$ can contribute, giving

$$
|A^{(L,L)}(\epsilon)\rangle = \sum_{\ell_0=0}^{L} \frac{1}{(L-\ell_0)!} f_1^{L-\ell_0}|H^{(\ell_0,\ell_0)}(\epsilon)\rangle, \qquad \text{where} \quad f_1 = \frac{1}{\epsilon}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad (3.18)
$$

exact to all orders in the ϵ expansion. Keeping just the first two terms in the Laurent expansion, we find

$$
\begin{aligned}
|A^{(L,L)}(\epsilon)\rangle &= \frac{1}{(L-1)!} f_1^{L-1} \left[\frac{1}{L} f_1 |A^{(0)}\rangle + |H^{(1,1)}(\epsilon)\rangle \right] + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right) \\
&= \frac{1}{(L-1)!} \frac{1}{\epsilon^{L-1}} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^{L-1} |A^{(1,1)}(L\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right).
\end{aligned}
\tag{3.19}
$$

This confirms the conjecture made in eqs. (4.45) and (4.46) of ref. [27].

3.3 IR divergences of $A^{(L,1)}$

In this section, we consider the subleading-color amplitude $|A^{(L,1)}\rangle$, and derive the first three[9] terms in the Laurent expansion. Consider all terms in eq. (3.6) for which the expression in square brackets in eq. (3.9) is ≤ 2:

$$
\begin{aligned}
|A^{(L)}(\epsilon)\rangle = & \frac{1}{L!}\left(g_1 + \frac{1}{N}f_1\right)^L |A^{(0)}\rangle + \frac{1}{N(L-1)!}\left(g_1 + \frac{1}{N}f_1\right)^{L-1}|H^{(1,1)}(\epsilon)\rangle \tag{3.20} \\
& + \frac{1}{(L-2)!}\left(g_1 + \frac{1}{N}f_1\right)^{L-2}\left(g_2 + \frac{1}{N}f_2\right)|A^{(0)}\rangle \\
& + \frac{1}{(L-1)!}\left(g_1 + \frac{1}{N}f_1\right)^{L-1}|H^{(1,0)}(\epsilon)\rangle \\
& + \frac{1}{N^2(L-2)!}\left(g_1 + \frac{1}{N}f_1\right)^{L-2}|H^{(2,2)}(\epsilon)\rangle + \cdots \qquad \textit{(three leading IR poles)}
\end{aligned}
$$

where we use eqs. (3.3) and (3.4) to write

$$
\begin{aligned}
g_1 &= \left(\frac{\mu^2}{Q^2}\right)^\epsilon \left[-\frac{2}{\epsilon^2}\mathbb{1} + \frac{1}{\epsilon}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right], & f_1 &= \frac{1}{\epsilon}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \\
g_2 &= \left(\frac{\mu^2}{Q^2}\right)^{2\epsilon} \left[-\left(\frac{\gamma^{(2)}}{8\epsilon^2} + \frac{\mathcal{G}_0^{(2)}}{2\epsilon}\right)\mathbb{1} + \frac{\gamma^{(2)}}{8\epsilon}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right], & f_2 &= \frac{\gamma^{(2)}}{8\epsilon}\left(\frac{\mu^2}{Q^2}\right)^{2\epsilon} \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.
\end{aligned}
\tag{3.21}
$$

To extract the $|A^{(L,1)}\rangle$ amplitude, we employ the identity

$$
\left.\left(g_1 + \frac{1}{N}f_1\right)^L\right|_{1/N \text{ piece}} \tag{3.22}
$$
$$
= Lg_1^{L-1}f_1 + \binom{L}{2}g_1^{L-2}[f_1,g_1] + \binom{L}{3}g_1^{L-3}[[f_1,g_1],g_1] + [\cdots[[[f_1,g_1],g_1],g_1]\cdots]
$$

in which the first term on the r.h.s. has an expansion that starts with $1/\epsilon^{2L-1}$, the second term has an expansion that starts with $1/\epsilon^{2L-2}$, and so forth. Thus, keeping only the terms proportional to $1/N$ in eq. (3.20), and only the first three terms in the Laurent expansion,

---

[9]It is straightforward to obtain further terms in the Laurent expansion as needed.

we obtain

$$
\begin{aligned}
|A^{(L,1)}\rangle = {} & \frac{1}{(L-1)!} g_1^{L-1} f_1 |A^{(0)}\rangle + \frac{1}{2(L-2)!} g_1^{L-2} [f_1, g_1] |A^{(0)}\rangle + \frac{1}{(L-1)!} g_1^{L-1} |H^{(1,1)}(\epsilon)\rangle \\
& + \frac{1}{6(L-3)!} g_1^{L-3} [[f_1, g_1], g_1] |A^{(0)}\rangle + \frac{1}{(L-2)!} g_1^{L-2} f_2 |A^{(0)}\rangle \\
& + \frac{1}{(L-3)!} g_1^{L-3} f_1 g_2 |A^{(0)}\rangle + \frac{1}{(L-2)!} g_1^{L-2} f_1 |H^{(1,0)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2L-4}}\right). \quad (3.23)
\end{aligned}
$$

In section 5.3, we will study the subleading-color amplitude $|A^{(L,1)}\rangle$ in the Regge limit $s \gg -t$, with $t < 0$ held fixed. In anticipation of that, we now compute the Regge limit of the IR-divergent expression (3.23), neglecting terms suppressed by powers of t/s. It is convenient in the Regge limit to choose the factorization scale Q^2 equal to the (fixed) momentum scale $-t$. Thus, using eq. (3.21) together with eqs. (B.1) and (B.8) we obtain

$$
\begin{aligned}
|A^{(L,1)}(\epsilon)\rangle = {} & \left(\frac{-4iK}{st}\right) \left(\frac{\mu^2}{-t}\right)^{L\epsilon} \frac{(-2)^L}{(L-1)!} \frac{Y}{\epsilon^{2L-1}} \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3(L-1)}{4}\epsilon \begin{pmatrix} -Z \\ X \\ 0 \end{pmatrix} \right. \\
& \left. + \frac{(L-1)(L-2)}{24}\epsilon^2 \begin{pmatrix} 7Z^2 \\ 7X^2 \\ XZ \end{pmatrix} + \frac{(L^2-17L+12)\zeta_2}{8}\epsilon^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}(\epsilon^3) + \mathcal{O}\left(t/s\right) \right]
\end{aligned} \quad (3.24)
$$

where we have suppressed the first six (vanishing) entries of the vector. To obtain eq. (3.24), we also needed to use terms through $\mathcal{O}(\epsilon)$ in

$$
|H^{(1,1)}(\epsilon)\rangle = \left(\frac{-4iK}{st}\right) \left[\zeta_2 Y \epsilon + \mathcal{O}(\epsilon^2) + \mathcal{O}\left(t/s\right)\right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.25)
$$

as well as the $\epsilon \to 0$ limit of $|H^{(1,0)}(\epsilon)\rangle$, namely

$$
|H^{(1,0)}(0)\rangle = \left(\frac{-4iK}{st}\right) \left[4\zeta_2 \left(1, 0, -1, -1, 0, 1, 0, 0, 0\right)^T + \mathcal{O}\left(t/s\right)\right] \quad (3.26)
$$

which are obtained from the Laurent expansions of the exact expressions (4.9) and (4.18). (Note that terms suppressed by powers of t/s have been omitted in both eqs. (3.25) and (3.26).)

4 Regge limit of $\mathcal{N} = 4$ SYM four-gluon amplitudes

In this section, we consider the leading logarithmic behavior of L-loop planar and subleading-color $\mathcal{N} = 4$ SYM four-gluon amplitudes in the Regge limit $s \gg -t$, with $t < 0$ held fixed. In section 5, we sum the leading logs to obtain the Regge trajectories.

4.1 Expectations from transcendentality

The L-loop planar and subleading-color amplitudes may be written as

$$|A^{(L,k)}(\epsilon)\rangle = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{L\epsilon}\sum_{m=-2L+k}^{\infty}\epsilon^m |a_m^{(L,k)}(s/t)\rangle \tag{4.1}$$

where $|a_m^{(L,k)}(s/t)\rangle$ is generally a complicated function of logarithms and polylogarithms. (We consider the amplitude in the physical region $s > 0$ and $t, u < 0$, with $s + t + u = 0$.) All $\mathcal{N} = 4$ SYM amplitudes have been observed to have uniform transcendentality [27, 34, 41]. This means that $|a_m^{(L,k)}(s/t)\rangle$ is a function of s/t whose degree of transcendentality[10] is $2L + m$.

Now we consider $|a_m^{(L,k)}(s/t)\rangle$ in the Regge limit $s \gg -t$, with $t < 0$ held fixed. Dropping any terms suppressed by at least one power of t/s, we are left with a polynomial in $\log(-s/t)$. Since logarithms have unit transcendentality, the degree of the polynomial can be no greater than $2L + m$. In the Regge limit, $|a_m^{(L,k)}(s/t)\rangle$ will be dominated by the leading term in the polynomial. *A priori* we might expect this term to be the maximum allowed by transcendentality, so that

$$|a_m^{(L,k)}(s/t)\rangle \xrightarrow[s \gg -t]{} \text{const}\left[\log\left(-\frac{s}{t}\right)\right]^{2L+m} + \text{subleading} \qquad (\textit{a priori expectation}) \tag{4.2}$$

where "subleading" indicates that we have dropped lower powers of $\log(-s/t)$ as well as terms suppressed by powers of t/s.

The expectation (4.2), however, is incorrect; the leading power of $\log(-s/t)$ is almost always less than the maximum allowed by transcendentality. The evidence suggests that the Regge limit of the planar L-loop amplitude is given by[11]

$$|a_m^{(L,0)}(s/t)\rangle \xrightarrow[s \gg -t]{} c_{L+m}\left[\log\left(-\frac{s}{t}\right)\right]^L + \text{subleading} \quad (\textit{conjecture}) \tag{4.3}$$

where c_{L+m} is a constant with degree of transcendentality $L+m$ (and vanishes for $m < -L$, in which case the lower powers of $\log(-s/t)$ cannot be neglected). The leading logarithmic growth of subleading-color amplitudes in the Regge limit appears to be even weaker than that for planar amplitudes, and we conjecture that

$$|a_m^{(L,k)}(s/t)\rangle \xrightarrow[s \gg -t]{} c'_{L+m+1}\left[\log\left(-\frac{s}{t}\right)\right]^{L-1} + \text{subleading}, \quad \text{for } k \geq 1 \quad (\textit{conjecture}) \tag{4.4}$$

where c'_{L+m+1} is a constant with degree of transcendentality $L+m+1$ (and vanishes when $m < -L-1$). We will discuss the evidence for the claims (4.3) and (4.4) in the remainder of this section.

---

[10]Each factor of ζ_k, π^k, $\log^k(-s/t)$, or any polylogarithm of total degree k has transcendentality k, and the transcendentality of a product of factors is additive.

[11]Terms suppressed by powers of t/s, however, can, and do, contain powers of $\log(-s/t)$ higher than L.

4.2 Regge limit of planar amplitudes

In this section, we review the Regge limit of the BDS ansatz for the planar four-gluon amplitude, which was explored in refs. [8, 17, 18].

The BDS ansatz for $A_{[1]}^{(L,0)}$ is [6]

$$A_{[1]}^{(L,0)} = M^{(L)}(s,t;\epsilon) \, A_{[1]}^{(0)}, \tag{4.5}$$

$$A_{[1]}^{(0)} = -\frac{4iK}{st}, \tag{4.6}$$

$$1 + \sum_{L=1}^{\infty} a^L M^{(L)}(s,t;\epsilon) = \exp\left\{\sum_{\ell=1}^{\infty} a^\ell \left[f^{(\ell)}(\epsilon) M^{(1)}(s,t;\ell\epsilon) + h^{(\ell)}(s,t;\epsilon)\right]\right\} \tag{4.7}$$

where

$$f^{(\ell)}(\epsilon) = \frac{1}{4}\gamma^{(\ell)} + \frac{1}{2}\epsilon\,\ell\,\mathcal{G}_0^{(\ell)} + \epsilon^2 f_2^{(\ell)} \tag{4.8}$$

with $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$ defined in eq. (2.8), and $h^{(\ell)}(s,t;\epsilon)$, which is finite as $\epsilon \to 0$, contains information about the short-distance behavior of the amplitude. The ratio of the one-loop amplitude to the tree amplitude is

$$M^{(1)}(s,t;\epsilon) = -\frac{1}{2}st\,I_4^{(1)}(s,t) \tag{4.9}$$

where the scalar box integral

$$I_4^{(1)}(s,t) = -i\mu^{2\epsilon}\mathrm{e}^{\epsilon\gamma}\pi^{-D/2}\int \frac{d^D p}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2} \tag{4.10}$$

may be evaluated exactly in terms of the hypergeometric function [42]. The BDS conjecture (4.7) for the four-gluon amplitude is wholly consistent with the IR-divergence structure as reviewed in section 2 and appendix A, but goes beyond it to assert that $h^{(\ell)}(s,t;\epsilon)$ is independent of s and t in the limit $\epsilon \to 0$.

In the Regge limit $s \gg -t$, one finds [18, 43], neglecting terms suppressed by $\mathcal{O}(t/s)$,

$$M^{(1)}(s,t;\epsilon) = \left(\frac{\mu^2}{-t}\right)^\epsilon \frac{r(\epsilon)}{\epsilon}\left[\log\left(-\frac{s}{t}\right) - i\pi + \psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)\right] + \mathcal{O}\left(t/s\right)$$

$$= \left(\frac{\mu^2}{-t}\right)^\epsilon r(\epsilon)\left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\log\left(-\frac{s}{t}\right) - \frac{i\pi}{\epsilon} + \sum_{m=0}^{\infty}[2+(-1)^m]\zeta_{m+2}\epsilon^m\right] + \mathcal{O}\left(t/s\right) \tag{4.11}$$

where

$$r(\epsilon) = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\mathrm{e}^{\gamma\epsilon} = 1 - \frac{1}{2}\zeta_2\epsilon^2 - \frac{7}{3}\zeta_3\epsilon^3 - \frac{47}{16}\zeta_4\epsilon^4 + \cdots \tag{4.12}$$

If the $h^{(\ell)}(s,t;\epsilon)$ term were absent from eq. (4.7), then eq. (4.11) would suffice to establish that $A^{(L,0)}$ goes as $\log^L(-s/t)$ in the Regge limit, as claimed in eq. (4.3). This claim would still be valid, even with the $h^{(\ell)}(s,t;\epsilon)$ term present, provided that $h^{(\ell)}(s,t;\epsilon)$ grows no

faster than $\log^\ell(-s/t)$ in the Regge limit.[12] In fact, the situation may be better than this. Using the explicit expressions in ref. [6] together with the help of the Mathematica package HPL [44] we find that

$$
h^{(2)}(s,t;\epsilon) = -\frac{\pi^4}{72} + \left(-\frac{11\pi^4}{360} \left[\log\left(-\frac{s}{t} \right) - i\pi \right] - \frac{39}{2}\zeta_5 + \frac{23\pi^2}{12}\zeta_3 \right)\epsilon \tag{4.13}
$$
$$
+ \left(\left[\frac{41}{2}\zeta_5 + \frac{\pi^2}{4}\zeta_3 \right] \left[\log\left(-\frac{s}{t} \right) - i\pi \right] - 15\zeta_3^2 - \frac{1789\pi^6}{30240} \right)\epsilon^2 + \mathcal{O}(\epsilon^3) + \mathcal{O}(t/s)
$$

so that $h^{(2)}(s,t;\epsilon)$ only grows as $\log(-s/t)$, at least to $\mathcal{O}(\epsilon^2)$. If we make the assumption that $h^{(\ell)}(s,t;\epsilon)$ grows less strongly than $\log^\ell(-s/t)$ in the Regge limit for all ℓ, then it would make no contribution to the leading log behavior of the planar L-loop amplitude, and we could conclude that[13]

$$
A_{[1]}^{(L,0)} \quad \underset{s \gg -t}{\longrightarrow} \quad \frac{1}{L!} \left(\frac{-4iK}{st} \right) \left(\frac{\mu^2}{-t} \right)^{L\epsilon} \left(\frac{r(\epsilon)}{\epsilon} \right)^L \left[\log\left(-\frac{s}{t} \right) \right]^L + \text{subleading} \tag{4.14}
$$

This behavior is precisely in accord with eq. (4.3), with $(r(\epsilon)/\epsilon)^L$ yielding constants c_{L+m} with the expected degree of transcendentality.

Now we consider the Regge limits of the other color-ordered amplitudes[14]

$$
A_{[2]}^{(L,0)} = M^{(L)}(s,u;\epsilon)\, A_{[2]}^{(0)}, \qquad A_{[3]}^{(L,0)} = M^{(L)}(t,u;\epsilon)\, A_{[3]}^{(0)}. \tag{4.15}
$$

These are also given by the BDS ansatz. To obtain $A_{[3]}^{(L,0)}$ we replace $\log(-s/t) - i\pi$ with $\log(u/t) = \log(-s/t) + \mathcal{O}(t/s)$ in eq. (4.11) to obtain[15]

$$
M^{(1)}(t,u;\epsilon) = \left(\frac{\mu^2}{-t} \right)^\epsilon r(\epsilon) \left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\log\left(-\frac{s}{t} \right) + \sum_{m=0}^\infty [2 + (-1)^m]\zeta_{m+2}\epsilon^m \right] + \mathcal{O}\left(t/s \right) \tag{4.16}
$$

Then, again subject to the assumption about $h^{(\ell)}(s,t;\epsilon)$ made above, $A_{[3]}^{(L,0)}$ also has leading log behavior in the Regge limit given by eq. (4.14).

On the other hand, $M^{(1)}(s,u;\epsilon)$ grows faster than $\log(-s/t)$ in the Regge limit,

$$
M^{(1)}(s,u;\epsilon) = \left(\frac{\mu^2}{-t} \right)^\epsilon \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\log\left(-\frac{s}{t} \right) - \frac{i\pi}{\epsilon} - \log^2\left(-\frac{s}{t} \right) + i\pi\log\left(-\frac{s}{t} \right) + 4\zeta_2 + \mathcal{O}(\epsilon) \right]
$$
$$
+ \mathcal{O}\left(t/s \right), \tag{4.17}
$$

---

[12]The $h^{(\ell)}(s,t;\epsilon)$ can affect the coefficients of nonpositive powers of ϵ in $A^{(L,0)}$ through interference with the IR-divergent terms in $M^{(1)}(s,t;\ell\epsilon)$.

[13]Interestingly, the individual scalar L-loop diagrams that contribute to the planar L-loop amplitude generically behave as eq. (4.2) in the Regge limit, but all powers of $\log(-s/t)$ higher than L cancel when they are added up.

[14]Also, recall that $A_{[4]}^{(L,0)} = A_{[3]}^{(L,0)}$, $A_{[5]}^{(L,0)} = A_{[2]}^{(L,0)}$, and $A_{[6]}^{(L,0)} = A_{[1]}^{(L,0)}$.

[15]Terms which are subleading in t/s can in principle lead to subleading Regge trajectories and/or cuts, which we do not examine in this paper. The terms of $\mathcal{O}(t/s)$ relative to the terms we keep could in principle lead to Regge trajectories passing through $j = 0$ at $t = 0$. This possibility is investigated in ref. [45].

and so $M^{(L)}(s, u; \epsilon)$ grows faster than $\log^L(-s/t)$. This apparent contradiction to eq. (4.3) is resolved by recognizing that $A_{[2]}^{(L,0)}$ is suppressed by t/s relative to $A_{[1]}^{(L,0)}$ and $A_{[3]}^{(L,0)}$, because $A_{[2]}^{(0)} = -4iK/su$, and is therefore entirely contained in the "subleading" term. In addition, the $-\log^2(-s/t)$ dependence in eq. (4.17) will lead to exponential suppression of the Regge trajectory associated with this amplitude, as we will see in section 5.

4.3 Regge limit of $A^{(1,1)}$

In this paper, we are particularly interested in the Regge behavior of subleading-color amplitudes. The simplest case is the one-loop subleading-color amplitude, which is given by [46]

$$A_{[7]}^{(1,1)} = A_{[8]}^{(1,1)} = A_{[9]}^{(1,1)} = 2\left(A_{[1]}^{(1,0)} + A_{[2]}^{(1,0)} + A_{[3]}^{(1,0)}\right). \tag{4.18}$$

We use eqs. (4.7), (4.11), (4.15), and (4.16), and recall that $A_{[2]}^{(1,0)}$ is suppressed by t/s, to obtain, in the Regge limit,

$$\begin{aligned}
A_{[7]}^{(1,1)} &= \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^\epsilon \left[-\frac{2\pi i r(\epsilon)}{\epsilon} + \mathcal{O}(t/s)\right] \tag{4.19} \\
&= \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^\epsilon \left[-\frac{2\pi i}{\epsilon} + i\pi\zeta_2\epsilon + \frac{14\pi i}{3}\zeta_3\epsilon^2 + \frac{47\pi i}{8}\zeta_4\epsilon^3 + \cdots + \mathcal{O}(t/s)\right].
\end{aligned}$$

This confirms the conjectured behavior (4.4) in the case $L = k = 1$. It was previously shown in eq. (40) of ref. [47] that the real part of $A_{[7]}^{(1,1)}$ vanishes to $\mathcal{O}(t/s)$.

4.4 Regge limits of $A^{(2,1)}$ and $A^{(2,2)}$

The two-loop subleading-color amplitudes $|A^{(2,1)}\rangle$ and $|A^{(2,2)}\rangle$ are known exactly [23]. The former is given by

$$\begin{aligned}
A_{[7]}^{(2,1)} &= -2iK\Big[s\left(3I_4^{(2)P}(s,t) + 2I_4^{(2)NP}(s,t) + 3I_4^{(2)P}(s,u) + 2I_4^{(2)NP}(s,u)\right) \tag{4.20} \\
&\quad -t\left(I_4^{(2)NP}(t,s) + I_4^{(2)NP}(t,u)\right) - u\left(I_4^{(2)NP}(u,s) + I_4^{(2)NP}(u,t)\right)\Big]
\end{aligned}$$

where $A_{[8]}^{(2,1)}$ and $A_{[9]}^{(2,1)}$ may be obtained via cyclic permutations of s, t, and u. The two-loop planar and non-planar scalar integrals appearing in eq. (4.20) are

$$\begin{aligned}
I_4^{(2)P}(s,t) &= \left(-i\mu^{2\epsilon}e^{\epsilon\gamma}\pi^{-D/2}\right)^2 \times \tag{4.21} \\
&\quad \times \int \frac{d^D p\, d^D q}{p^2\,(p+q)^2 q^2\,(p-k_1)^2\,(p-k_1-k_2)^2\,(q-k_4)^2\,(q-k_3-k_4)^2} \\
I_4^{(2)NP}(s,t) &= \left(-i\mu^{2\epsilon}e^{\epsilon\gamma}\pi^{-D/2}\right)^2 \times \\
&\quad \times \int \frac{d^D p\, d^D q}{p^2\,(p+q)^2\,q^2\,(p-k_2)^2\,(p+q+k_1)^2\,(q-k_3)^2\,(q-k_3-k_4)^2}.
\end{aligned}$$

Explicit expressions for these integrals are given in refs. [6] and [25] respectively. Continuing these expressions to the physical region $s > 0$, $t, u < 0$, and dropping terms suppressed by

t/s in the Regge limit, we obtain

$$A_{[7]}^{(2,1)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon} \left\{ \frac{4i\pi}{\epsilon^3} - \frac{3i\pi\left[\log(-s/t) - i\pi\right]}{\epsilon^2} - \frac{3i\pi^3}{2\epsilon} \right.$$
$$\left. + \frac{i\pi}{6}\left[3\pi^2\log(-s/t) - 82\zeta_3 - 3i\pi^3\right] + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}$$

$$A_{[8]}^{(2,1)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon} \left\{ \frac{4i\pi}{\epsilon^3} - \frac{3i\pi\log(-s/t)}{\epsilon^2} - \frac{3i\pi^3}{2\epsilon} \right.$$
$$\left. + \frac{i\pi}{6}\left[3\pi^2\log(-s/t) - 82\zeta_3\right] + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}$$

$$A_{[9]}^{(2,1)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon} \left\{ \frac{4i\pi}{\epsilon^3} - \frac{3i\pi^3}{2\epsilon} - \frac{95}{3}i\pi\zeta_3 + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}. \tag{4.22}$$

By using eq. (B.9), one may easily verify that that IR-divergent parts of this expression agree with the general expression (3.24) derived in the last section.

The most-subleading-color two-loop amplitudes are given by [23].

$$A_{[1]}^{(2,2)} = -2iK\left[s\left(I_4^{(2)P}(s,t) + I_4^{(2)NP}(s,t) + I_4^{(2)P}(s,u) + I_4^{(2)NP}(s,u)\right)\right.$$
$$+ t\left(I_4^{(2)P}(t,s) + I_4^{(2)NP}(t,s) + I_4^{(2)P}(t,u) + I_4^{(2)NP}(t,u)\right) \tag{4.23}$$
$$\left. - 2u\left(I_4^{(2)P}(u,s) + I_4^{(2)NP}(u,s) + I_4^{(2)P}(u,t) + I_4^{(2)NP}(u,t)\right)\right].$$

The other single-trace amplitudes $A_{[i]}^{(2,2)}$ are obtained by making the appropriate permutations of s, t, and u in this expression. Again extracting the Regge limit of these amplitudes, we find

$$A_{[1]}^{(2,2)} = A_{[6]}^{(2,2)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon} \tag{4.24}$$
$$\times\left\{ \frac{i\pi\left[\log(-s/t) + i\pi\right]}{\epsilon^2} - \frac{i\pi}{6}\left[\pi^2\log(-s/t) + i\pi^3 + 36\zeta_3\right] + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}$$

$$A_{[2]}^{(2,2)} = A_{[5]}^{(2,2)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon}$$
$$\times\left\{ \frac{i\pi\left[-2\log(-s/t) + i\pi\right]}{\epsilon^2} + \frac{i\pi}{6}\left[2\pi^2\log(-s/t) - i\pi^3 + 72\zeta_3\right] + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}$$

$$A_{[3]}^{(2,2)} = A_{[4]}^{(2,2)} = \left(\frac{-4iK}{st}\right)\left(\frac{\mu^2}{-t}\right)^{2\epsilon}$$
$$\times\left\{ \frac{i\pi\left[\log(-s/t) - 2\pi i\right]}{\epsilon^2} - \frac{i\pi}{6}\left[\pi^2\log(-s/t) - 2i\pi^3 + 36\zeta_3\right] + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(t/s\right) \right\}.$$

We see that both subleading-color amplitudes $|A^{(2,1)}\rangle$ and $|A^{(2,2)}\rangle$ go as $\log(-s/t)$ in the Regge limit (at least through $\mathcal{O}(\epsilon^0)$), thus adding support to our conjecture (4.4).

4.5 Regge limit of IR-divergences of higher-loop amplitudes

In section 4.3, we saw that the one-loop subleading-color amplitude goes as $\log^0(-s/t)$ to all orders in ϵ, and in section 4.4 that all two-loop subleading-color amplitudes go as $\log^1(-s/t)$, at least through $\mathcal{O}(\epsilon^0)$. Thus suggests that, while the L-loop *planar* amplitude (probably) goes as $\log^L(-s/t)$ in the Regge limit, the L-loop *subleading-color* amplitudes only go as $\log^{L-1}(-s/t)$ in the Regge limit, as conjectured in eq. (4.4).

Because the IR-finite parts of the subleading-color amplitudes beyond two loops are not known explicitly, we cannot prove this conjecture, but in this section we will perform an important consistency check. We will prove that the *IR-divergent* contributions to the L-loop subleading-color amplitudes grow no faster than $\log^{L-1}(-s/t)$ in the Regge limit, provided that ℓ_0-loop subleading-color amplitudes (both IR-divergent and finite parts) grow no faster than $\log^{\ell_0-1}(-s/t)$ for all $\ell_0 < L$. Thus, with this inductive argument, it is sufficient to prove that the *IR-finite* contribution to the L-loop subleading-color amplitudes goes as $\log^{L-1}(-s/t)$ to establish it for the full amplitude.

Our first step is to prove a weaker result, namely that the IR-divergent part of any L-loop amplitude (planar or subleading) grows no faster than $\log^L(-s/t)$, provided that no ℓ_0-loop amplitude (planar or subleading) with $\ell_0 < L$ grows faster than $\log^{\ell_0}(-s/t)$. Consider $\mathbf{G}^{(\ell)}/N^\ell$ defined by eqs. (3.3) and (3.4), with $Q^2 = -t$, and α through δ given by eq. (B.8). In eq. (B.10), we show that α through δ, and therefore $\mathbf{G}^{(\ell)}/N^\ell$, go as $\log(-s/t)$ in the Regge limit. Consequently, the strongest growth of any (IR-divergent) term in eq. (3.2) is $\log^q(-s/t)$ where $q = \ell_0 + \sum_{\ell=1}^{\infty} n_\ell$. Since $q \leq L$ by eq. (3.7), we have established our result.

Now we prove a stronger result, namely that the $\log^L(-s/t)$ terms are actually absent from the IR-divergent contributions to L-loop *subleading-color* amplitudes. The only terms in eq. (3.6) that could yield $\log^L(-s/t)$ growth are those with $n_\ell = 0$ for $\ell > 1$ (so that the inequality $q \leq L$ is saturated) and containing $|H^{(\ell_0,0)}\rangle$ (since we assume that ℓ_0-loop subleading-color amplitudes grow no faster than $\log^{\ell_0-1}(-s/t)$), namely, terms of the form

$$\left(g_1 + \frac{1}{N}f_1\right)^{L-\ell_0} |H^{(\ell_0,0)}(\epsilon)\rangle. \tag{4.25}$$

First we consider the $\ell_0 = 0$ term

$$\left(g_1 + \frac{1}{N}f_1\right)^{L} |A^{(0)}\rangle \tag{4.26}$$

with $|A^{(0)}\rangle$ given in the Regge limit by

$$|A^{(0)}\rangle \xrightarrow[s \gg -t]{} -\frac{4iK}{st}(1,0,-1,-1,0,1,0,0,0)^T. \tag{4.27}$$

Since any subleading-color amplitude contains at least one factor of f_1, we can see that the structure $\cdots \gamma \alpha^n |A^{(0)}\rangle$ for some $n \geq 0$ will always appear. By virtue of eqs. (4.27) and (B.10), one can see that the leading log term in $\cdots \gamma \alpha^n |A^{(0)}\rangle$ vanishes since α doesn't change the structure of $|A^{(0)}\rangle$ and γ annihilates it.

Essentially the same argument works for the $\ell_0 \neq 0$ terms as well. By the BDS ansatz, the leading-color amplitudes $|A^{(\ell_0,0)}(\epsilon)\rangle$ are given by

$$A_{[1]}^{(\ell_0,0)} = M^{(\ell_0)}(s,t)A_{[1]}^{(0)}, \qquad A_{[3]}^{(\ell_0,0)} = M^{(\ell_0)}(u,t)A_{[3]}^{(0)}. \tag{4.28}$$

But the leading log terms of $M^{(\ell_0)}(u,t)$ and $M^{(\ell_0)}(s,t)$ are equal in the Regge limit, so the leading log piece of $|A^{(\ell_0,0)}(\epsilon)\rangle$ (and therefore of the IR-finite contribution $|H^{(\ell_0,0)}(\epsilon)\rangle$) is proportional to $|A^{(0)}\rangle$. Thus the putative $\log^L(-s/t)$ terms of $\cdots \gamma\alpha^n|H^{(\ell_0,0)}\rangle$ also vanish.

Hence, we conclude that the IR-divergent terms of the L-loop subleading-color amplitudes go as $\log^{L-1}(-s/t)$, provided that the same holds for all lower-loop subleading-color amplitudes.

5 Regge trajectories

In section 4, we discussed the leading log behavior of the L-loop planar and subleading-color amplitudes in the Regge limit. In this section, we will sum the loop amplitudes to obtain Regge trajectories.

5.1 Planar gluon Regge trajectory

We first focus on the planar color-ordered amplitude $A_{[3]}$, which is real in the region $s > 0$, $t, u < 0$, and whose Regge behavior was explored in refs. [8, 17, 18]. The $\log^L(-s/t)$ behavior of the Regge limit of the planar L-loop amplitude $A_{[3]}^{(L,0)}$ conjectured in section 4 suggests that the all-orders planar amplitude exhibits Regge behavior

$$\sum_{L=0}^{\infty} a^L A_{[3]}^{(L,0)} \quad \underset{s \gg -t}{\longrightarrow} \quad \beta_0(t)\left(-\frac{s}{t}\right)^{\alpha_0(t)} \tag{5.1}$$

where $\alpha_0(t)$ is the Regge trajectory function, and $\beta_0(t)$ the Regge residue. Indeed, using eqs. (4.7), (4.15), and (4.16), one obtains the following expression for the Regge trajectory function [18]

$$\alpha_0(t) = 1 + \sum_{\ell=1}^{\infty} f^{(\ell)}(\epsilon)\frac{r(\ell\epsilon)}{\ell\epsilon}a^\ell\left(\frac{\mu^2}{-t}\right)^{\ell\epsilon} + \mathcal{O}(\epsilon). \tag{5.2}$$

where $\mathcal{O}(\epsilon)$ corrections come from the $h^{(\ell)}(s,t;\epsilon)$ terms[16] in eq. (4.7). For example, the $\log(-s/t)$ dependent terms in eq. (4.13) contribute to the two-loop Regge trajectory [18] at $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$. The leading 1 comes from the tree amplitude $A_{[3]}^{(0)} = -4iK/ut$ since

$$-4iK \quad \underset{s \gg -t}{\longrightarrow} \quad ks^2 \tag{5.3}$$

where k depends on the helicities of the gluons, and is finite as $s \to \infty$. We can rewrite eq. (5.2) as [8, 17]

$$\begin{aligned}
\alpha_0(t) &= 1 + \frac{1}{4\epsilon}\sum_{\ell=1}^{\infty}a^\ell\frac{\gamma^{(\ell)}}{\ell}\left(\frac{\mu^2}{-t}\right)^{\ell\epsilon} + \frac{1}{2}\sum_{\ell=1}^{\infty}a^\ell\mathcal{G}_0^{(\ell)} + \mathcal{O}(\epsilon) \\
&= 1 + \frac{1}{4\epsilon}\gamma^{(-1)}(a) - \frac{1}{4}\gamma(a)\log\left(\frac{-t}{\mu^2}\right) + \frac{1}{2}\mathcal{G}_0(a) + \mathcal{O}(\epsilon). \tag{5.4}
\end{aligned}$$

---

[16] In fact, Regge behavior (5.1) will hold to all orders in ϵ only if $h^{(\ell)}(s,t;\epsilon)$ grows no faster than $\log(-s/t)$ for all ℓ.

The residue is given by [17]

$$\beta_0(t) = k \exp\left\{ -\frac{1}{2\epsilon^2}\gamma^{(-2)}\left(\frac{\mu^{2\epsilon}a}{(-t)^\epsilon}\right) - \frac{1}{\epsilon}\mathcal{G}_0^{(-1)}\left(\frac{\mu^{2\epsilon}a}{(-t)^\epsilon}\right) + \zeta_2\gamma(a) - 2f_2^{(-2)}(a) + h(a) + \mathcal{O}(\epsilon) \right\}$$
(5.5)

where the functions in eqs. (5.4) and (5.5) are defined by

$$\gamma^{(-1)}(a) = \sum_{\ell=1}^{\infty}\frac{a^\ell}{\ell}\gamma^{(\ell)}, \qquad \gamma^{(-2)}(a) = \sum_{\ell=1}^{\infty}\frac{a^\ell}{\ell^2}\gamma^{(\ell)},$$
(5.6)

$$\mathcal{G}_0^{(-1)}(a) = \sum_{\ell=1}^{\infty}\frac{a^\ell}{\ell}\mathcal{G}_0^{(\ell)}, \qquad f_2^{(-2)}(a) = \sum_{\ell=1}^{\infty}\frac{a^\ell}{\ell^2}f_2^{(\ell)}, \qquad h(a) = \sum_{\ell=1}^{\infty}a^\ell h^{(\ell)}(0),$$

Explicitly, we have [6]

$$\zeta_2\gamma(a) - 2f_2^{(-2)}(a) + h(a) = 4\zeta_2 a - \frac{43}{4}\zeta_4 a^2 + \left(\frac{8657}{216}\zeta_6 - \frac{17}{9}\zeta_3^2\right)a^3 + \cdots$$
(5.7)

and $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$ are given in eq. (2.8).

We now consider the other color-ordered amplitudes $A_{[1]}$ and $A_{[2]}$. Using eq. (4.11), one can see that the planar contribution to $A_{[1]}$ goes in the Regge limit to

$$\sum_{L=0}^{\infty} a^L A_{[1]}^{(L,0)} \quad \xrightarrow[s\gg -t]{} \quad \beta_0(t)e^{-i\pi\alpha_0(t)}\left(-\frac{s}{t}\right)^{\alpha_0(t)}.$$
(5.8)

The presence of the $-\log^2(-s/t)$ term in eq. (4.17), however, results in the exponential suppression of $A_{[2]}$ in the Regge limit, $viz.$, $(-s/t)^{-\log(-s/t)} \to 0$.

We now rewrite the full planar four-gluon amplitude (2.1) as

$$\mathcal{A}_{4-\text{gluon}}^{\text{planar}} = g^2 \sum_{L=0}^{\infty} a^L \left[\left(A_{[1]}^{(L,0)} - A_{[3]}^{(L,0)}\right)f^{a_1a_4b}f^{a_2a_3b}\right.$$
(5.9)

$$\left. + \left(A_{[1]}^{(L,0)} + A_{[3]}^{(L,0)}\right)d^{a_1a_4b}d^{a_2a_3b} + A_{[2]}^{(L,0)}\left(\mathcal{C}_{[2]} + \mathcal{C}_{[5]}\right)\right]$$

where

$$f^{a_1a_4b}f^{a_2a_3b} = \frac{1}{2}\left(\mathcal{C}_{[1]} - \mathcal{C}_{[3]} - \mathcal{C}_{[4]} + \mathcal{C}_{[6]}\right), \qquad d^{a_1a_4b}d^{a_2a_3b} = \frac{1}{2}\left(\mathcal{C}_{[1]} + \mathcal{C}_{[3]} + \mathcal{C}_{[4]} + \mathcal{C}_{[6]}\right).$$
(5.10)

The coefficient of the $f^{a_1a_4b}f^{a_2a_3b}$ term in eq. (5.9) corresponds to the exchange of a trajectory in the t channel with the quantum numbers of the gluon, and so the planar gluon Regge trajectory is given by

$$\sum_{L=0}^{\infty} a^L \left(A_{[1]}^{(L,0)} - A_{[3]}^{(L,0)}\right) \quad \xrightarrow[s\gg -t]{} \quad B_0(t)\left(-\frac{s}{t}\right)^{\alpha_0(t)}$$
(5.11)

where $\alpha_0(t)$, given in eq. (5.4), represents the planar gluon Regge trajectory, and $B_0(t)$, given by $\beta_0(t)\left(e^{-i\pi\alpha_0(t)} - 1\right)$, is the Regge residue, including the signature factor. The coefficient of $d^{a_1a_4b}d^{a_2a_3b}$ in eq. (5.9) gives a wrong signature trajectory, and the $A_{[2]}$ term is exponentially damped in the Regge limit.

5.2 $1/N^2$ corrections

As seen in the previous section, the planar amplitudes sum up to give the planar gluon Regge trajectory (5.11). It might be expected that the full amplitude would give rise to subleading-color corrections to the gluon trajectory. Let us characterize the first subleading-color corrections to the gluon trajectory as

$$A_{[1]} - A_{[3]} \quad \xrightarrow{s \gg -t} \quad \left[B_0(t) + \frac{1}{N^2} B_2(t) + \cdots \right] \left(-\frac{s}{t} \right)^{\alpha_0(t) + (1/N^2)\alpha_2(t) + \cdots} \tag{5.12}$$

The $1/N^2$ corrections to the amplitude may be (at least partially) summed to give

$$\sum_{L=2}^{\infty} \frac{a^L}{N^2} A_{[1]}^{(L,2)} \to \exp\left[-\frac{2a}{\epsilon^2} + \cdots \right] \frac{a^2}{N^2} \left(\frac{\mu^2}{-t} \right)^{2\epsilon} \frac{i\pi ks}{t} \times \tag{5.13}$$

$$\times \left(\frac{\log(-s/t) + i\pi}{\epsilon^2} - \frac{\pi^2}{6} \log(-s/t) - \frac{i\pi^3}{6} - 6\zeta_3 \right)$$

$$\sum_{L=2}^{\infty} \frac{a^L}{N^2} A_{[3]}^{(L,2)} \to \exp\left[-\frac{2a}{\epsilon^2} + \cdots \right] \frac{a^2}{N^2} \left(\frac{\mu^2}{-t} \right)^{2\epsilon} \frac{i\pi ks}{t} \times \tag{5.14}$$

$$\times \left(\frac{\log(-s/t) - 2\pi i}{\epsilon^2} - \frac{\pi^2}{6} \log(-s/t) + \frac{i\pi^3}{3} - 6\zeta_3 \right)$$

where the IR-finite terms are obtained from the two-loop subleading-color amplitude (4.24), and the exponential prefactor results from summing the leading IR-divergent term (3.16) to all orders in L. All the $\log(-s/t)$ terms cancel from the combination of amplitudes that contributes to the gluon Regge trajectory

$$\sum_{L=2}^{\infty} \frac{a^L}{N^2} \left(A_{[1]}^{(L,2)} - A_{[3]}^{(L,2)} \right) \to \frac{a^2}{N^2} \left(\frac{\mu^2}{-t} \right)^{2\epsilon} \frac{ks}{t} \left(-\frac{3\pi^2}{\epsilon^2} + \frac{\pi^4}{2} \right) + \mathcal{O}(a^3) \tag{5.15}$$

and consequently, the gluon Regge trajectory function $\alpha_0(t)$ remains uncorrected through $\mathcal{O}(a^2)$, as might be anticipated from the corresponding two-loop result for QCD [35, 36]. The expression (5.15) corresponds to a $1/N^2$ correction

$$B_2(t) = ka^2 \left(\frac{\mu^2}{-t} \right)^{2\epsilon} \left(\frac{3\pi^2}{\epsilon^2} - \frac{\pi^4}{2} \right) + \mathcal{O}(a^3) \tag{5.16}$$

to the Regge residue starting at two loops [36].

5.3 Regge trajectory for double-trace amplitudes

In section 4, we presented evidence that the L-loop subleading-color amplitudes go as $\log^{L-1}(-s/t)$ in the Regge limit. This suggests that the double-trace amplitudes may also exhibit Regge behavior

$$\sum_{L=1}^{\infty} a^{L-1} |A^{(L,1)}\rangle \quad \xrightarrow{s \gg -t} \quad \beta_1(t) \left(-\frac{s}{t} \right)^{\alpha_1(t)}. \tag{5.17}$$

We will now see how far this expectation is borne out.

In section 3.3, we calculated the first three IR-divergent terms of the subleading-color amplitude $|A^{(L,1)}\rangle$. For the moment, let us focus on only one component

$$
\begin{aligned}
A_{[8]}^{(L,1)} &= \left(\frac{-4iK}{st}\right)\frac{Y\epsilon}{(L-1)!}\left[\frac{-2}{\epsilon^2}\left(\frac{\mu^2}{-t}\right)^\epsilon\right]^L\left\{1+\frac{3}{4}(L-1)X\epsilon\right.\\
&\qquad\left.+\frac{7}{24}(L-1)(L-2)X^2\epsilon^2+\frac{1}{8}(L^2-17L+12)\zeta_2\epsilon^2+\mathcal{O}(\epsilon^3)+\mathcal{O}(t/s)\right\}\\
&= \left(\frac{-4iK}{st}\right)\frac{-2Y}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon\frac{1}{(L-1)!}\left[\left(\frac{\mu^2}{-t}\right)^\epsilon\left(-\frac{2}{\epsilon^2}-\frac{3X}{2\epsilon}\right)\right]^{L-1}\\
&\qquad\times\left\{1+\frac{1}{96}(L-1)(L-2)X^2\epsilon^2+\frac{1}{8}(L^2-17L+12)\zeta_2\epsilon^2+\mathcal{O}(\epsilon^3)+\mathcal{O}(t/s)\right\}.
\end{aligned}
\tag{5.18}
$$

Since, by eq. (B.9), $X^2\gg1$ in the Regge limit, we can drop the ζ_2-dependent term in the curly braces in eq. (5.18). The series can be summed to obtain

$$
\begin{aligned}
\sum_{L=1}^\infty a^{L-1}A_{[8]}^{(L,1)} &= \left(\frac{-4iK}{st}\right)\frac{-2Y}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon\exp\left[\left(\frac{\mu^2}{-t}\right)^\epsilon\left(-\frac{2a}{\epsilon^2}-\frac{3aX}{2\epsilon}\right)\right]\left[1+\frac{a^2X^2}{24\epsilon^2}+\cdots\right]\\
&= \left(\frac{-4iK}{st}\right)\frac{-2Y}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon\exp\left[\left(\frac{\mu^2}{-t}\right)^\epsilon\left(-\frac{2a}{\epsilon^2}-\frac{3aX}{2\epsilon}\right)+\frac{a^2X^2}{24\epsilon^2}+\cdots\right]\\
&= \frac{2\pi ik}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon\exp\left[-\frac{2a}{\epsilon^2}\left(\frac{\mu^2}{-t}\right)^\epsilon\right]\left(\frac{s}{-t}\right)^{\alpha_1(t)\,+\,(a^2/24\epsilon^2)\log(-s/t)}
\end{aligned}
\tag{5.19}
$$

where in the last line of eq. (5.19) we used eqs. (5.3) and (B.9). The Regge trajectory function in eq. (5.19) is given by

$$
\alpha_1(t) = 1+\frac{3a}{2\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon = 1+\frac{3a}{2\epsilon}-\frac{3a}{2}\log\left(\frac{-t}{\mu^2}\right)+\cdots
\tag{5.20}
$$

Equation (5.20) suggests a massless spin-1 state with Regge slope $3/2$ that of the planar (gluon) trajectory. However, since eq. (5.19) cannot lead to a physical massless particle, we speculate that this is a trajectory which is nonsense-choosing[17] at $j=1$. By contrast, the gluon lies on a trajectory which chooses sense at $j=1$. The $a^2\log(-s/t)$ term in the exponent in eq. (5.19) can be interpreted as a Regge cut.

Starting from eq. (3.24), we obtain similar results in the Regge limit for $A_{[7]}^{(L,1)}$:

$$
\begin{aligned}
\sum_{L=1}^\infty a^{L-1}A_{[7]}^{(L,1)} &= \frac{2\pi ik}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon\exp\left[\left(\frac{\mu^2}{-t}\right)^\epsilon\left(-\frac{2a}{\epsilon^2}-\frac{3\pi ia}{2\epsilon}\right)-\frac{\pi^2a^2}{24\epsilon^2}\right]\\
&\qquad\times\left(\frac{s}{-t}\right)^{\alpha_1(t)-(i\pi a^2/12\epsilon^2)\,+\,(a^2/24\epsilon^2)\log(-s/t)}
\end{aligned}
\tag{5.21}
$$

---

[17] See section 5 of ref. [45] for a discussion of possible nonsense-choosing states in $\mathcal{N}=4$ SYM with gauge group SU(2). In that reference, trajectories with possible massless scalar bound states are also discussed, but not considered here, as these are $\mathcal{O}(t/s)$, and suppressed in the limits we consider.

while

$$\sum_{L=1}^{\infty} a^{L-1} A_{[9]}^{(L,1)} = \frac{2\pi i k}{\epsilon} \left(\frac{\mu^2}{-t}\right)^{\epsilon} \exp\left[-\frac{2a}{\epsilon^2}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\right] \left(\frac{s}{-t}\right)^{1 + (i\pi a^2/6\epsilon^2) \,-\, (a^2/6\epsilon^2)\log(-s/t)}$$

(5.22)

has a fixed pole together with a Regge cut, which leads to exponential damping.

6 Conclusions

Beginning with the assumption that all soft anomalous dimension matrices $\mathbf{\Gamma}^{(\ell)}$ are proportional to $\mathbf{\Gamma}^{(1)}$, and therefore commute with each other, we derived all-loop-order expressions for the IR-divergent parts of the planar and all subleading-color contributions to the $\mathcal{N} = 4$ SYM four-gluon amplitude. Explicit expressions for the leading IR divergences are presented in eqs. (3.15) and (3.16), confirming a conjecture of ref. [27]. The first two terms in the Laurent expansion in the IR regulator ϵ are presented for the most-subleading-color amplitude $A^{(L,L)}$ in eq. (3.19), also confirming a conjecture of ref. [27]. The three leading terms in the Laurent expansion in ϵ for $A^{(L,1)}$ are given in eq. (3.23), and their Regge limit in eq. (3.24); further terms in the Laurent expansion could be computed as needed.

The iterative structure of planar amplitudes was exploited in ref. [6] to formulate the BDS conjecture. No analogous results are known for subleading-color amplitudes. A weaker possibility is that the amplitude obtained by summing subleading-color amplitudes over all loops has Regge behavior in the limit $s \to \infty$, t fixed. (It is weaker because $\mathcal{O}(t/s)$ terms are neglected in this limit. In contrast, the planar four-gluon amplitude is Regge exact [8]; i.e., Regge behavior is manifest without taking any limit.) We first considered the Regge limit of four-gluon amplitudes, and presented evidence that the leading logarithmic growth of the subleading-color L-loop amplitudes is less severe than that of the planar amplitudes, going as $\log^{L-1}(-s/t)$ rather than $\log^L(-s/t)$. We then investigated $1/N^2$ corrections to the gluon Regge trajectory as well as Regge behavior of the subleading-color double-trace amplitudes by summing over the IR-divergent parts of the L-loop amplitudes, neglecting terms of $\mathcal{O}(t/s)$. The subleading-color double-trace amplitudes exhibit Regge behavior: that is, there is a Regge trajectory as well as a Regge cut which emerges at three loops. Thus, in the weaker sense described in this paper, there is sufficient iterative structure to produce leading Regge behavior in the subleading-color amplitudes.

Acknowledgments

A Generalized ABDK equation

In this appendix we show that the IR-divergent part of the L-loop generalization [6] of the ABDK relation [1] for the planar four-gluon amplitude may easily be obtained from the

expression (2.14) for the four-gluon amplitude

$$\sum_{L=0}^{\infty} a^L |A^{(L)}(\epsilon)\rangle = \exp\left[\sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} \mathbf{G}^{(\ell)}(\ell\epsilon)\right] \left(\sum_{L=0}^{\infty} a^L |H^{(L)}(\epsilon)\rangle\right) \tag{A.1}$$

Consider the planar (leading-color) L-loop amplitude $|A^{(L,0)}\rangle$, and its IR-finite part $|H^{(L,0)}\rangle$, which are proportional to the tree-level amplitude:

$$A_{[1]}^{(L,0)}(\epsilon) = M^{(L)}(\epsilon)A_{[1]}^{(0)}, \qquad H_{[1]}^{(L,0)}(\epsilon) = \tilde{M}^{(Lf)}(\epsilon)A_{[1]}^{(0)} \tag{A.2}$$

From the expressions (3.3), (3.4), and (B.4), we observe that the leading-color term of $\mathbf{G}^{(\ell)}$ is a diagonal matrix, and moreover that all subleading corrections are off-diagonal. Thus, retaining only the leading-color terms of eq. (A.1), we have

$$1 + \sum_{\ell=1}^{\infty} a^\ell M^{(\ell)}(\epsilon) = \exp\left[\sum_{\ell=1}^{\infty} \frac{a^\ell}{N^\ell} G_{[11]}^{(\ell)}(\ell\epsilon)\right] \left(1 + \sum_{\ell=1}^{\infty} a^\ell \tilde{M}^{(\ell f)}(\epsilon)\right) \tag{A.3}$$

where $G_{[11]}^{(\ell)}$ denotes the 11 matrix element of $\mathbf{G}^{(\ell)}$. Using eq. (2.17), we may rewrite this as

$$M^{(\ell)}(\epsilon) - X^{(\ell)}[M] = \frac{G_{[11]}^{(\ell)}(\ell\epsilon)}{N^\ell} + \tilde{M}^{(\ell f)}(\epsilon) - X^{(\ell)}[\tilde{M}^{(f)}] \tag{A.4}$$

which is valid to all orders in the ϵ expansion. Using eq. (3.3) we observe that

$$\begin{aligned}
\frac{G_{[11]}^{(\ell)}(\ell\epsilon)}{N^\ell} &= \frac{1}{2}\left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon}\left[-\frac{\gamma^{(\ell)}}{(\ell\epsilon)^2} - \frac{2\mathcal{G}_0^{(\ell)}}{\ell\epsilon} + \frac{\gamma^{(\ell)}}{4\ell\epsilon}\Gamma_{[11]}^{(1)}\right] \\
&= \left[\frac{\gamma^{(\ell)}}{4} + \frac{\ell}{2}\mathcal{G}_0^{(\ell)}\epsilon\right]\left(\frac{\mu^2}{Q^2}\right)^{\ell\epsilon}\left[-\frac{2}{(\ell\epsilon)^2} + \frac{\Gamma_{[11]}^{(1)}}{2\ell\epsilon}\right] + \mathcal{O}(\epsilon^0) \\
&= f^{(\ell)}(\epsilon)\frac{G_{[11]}^{(1)}(\ell\epsilon)}{N} + \mathcal{O}(\epsilon^0) \\
&= f^{(\ell)}(\epsilon)M^{(1)}(\ell\epsilon) + \mathcal{O}(\epsilon^0)
\end{aligned} \tag{A.5}$$

where $f^{(\ell)}(\epsilon)$ is defined in eq. (4.8). Hence we obtain

$$M^{(\ell)}(\epsilon) = X^{(\ell)}[M] + f^{(\ell)}(\epsilon)M^{(1)}(\ell\epsilon) + \mathcal{O}(\epsilon^0) \tag{A.6}$$

which is precisely the IR-divergent part of the generalized ABDK relation for the four-gluon amplitude, eq. (4.13) of ref. [6].

B Explicit expressions for the four-gluon amplitude

In this appendix we collect various explicit expressions for four-gluon amplitudes needed in the paper.

The tree-level amplitudes are

$$|A^{(0)}\rangle = -\frac{4iK}{stu}\,(u,t,s,s,t,u,0,0,0)^T \tag{B.1}$$

where s, t, and u are the Mandelstam invariants s_{12}, s_{14}, and s_{13}, where $s_{ij} = (k_i + k_j)^2$, with $s + t + u = 0$ for massless external gluons. The factor K, defined in eq. (7.4.42) of ref. [49], depends on the momenta and helicity of the external gluons, and is totally symmetric under permutations of the external legs.

The one-loop soft anomalous dimension matrix is given by [29]

$$\mathbf{\Gamma}^{(1)} = -\frac{1}{N}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j\,\log\left(\frac{-s_{ij}}{Q^2}\right) \tag{B.2}$$

where $\mathbf{T}_i\cdot\mathbf{T}_j = T_i^a T_j^a$ with T_i^a the SU(N) generators in the adjoint representation. In the basis (2.2), it has the explicit form [37]

$$\mathbf{\Gamma}^{(1)} = 2\begin{pmatrix} \alpha & \beta/N \\ \gamma/N & \delta \end{pmatrix} \tag{B.3}$$

where

$$\alpha = \begin{pmatrix} \mathcal{S}+\mathcal{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{S}+\mathcal{U} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{T}+\mathcal{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{T}+\mathcal{U} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{S}+\mathcal{U} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{S}+\mathcal{T} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathcal{T}-\mathcal{U} & 0 & \mathcal{S}-\mathcal{U} \\ \mathcal{U}-\mathcal{T} & \mathcal{S}-\mathcal{T} & 0 \\ 0 & \mathcal{T}-\mathcal{S} & \mathcal{U}-\mathcal{S} \\ 0 & \mathcal{T}-\mathcal{S} & \mathcal{U}-\mathcal{S} \\ \mathcal{U}-\mathcal{T} & \mathcal{S}-\mathcal{T} & 0 \\ \mathcal{T}-\mathcal{U} & 0 & \mathcal{S}-\mathcal{U} \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \mathcal{S}-\mathcal{U} & \mathcal{S}-\mathcal{T} & 0 & 0 & \mathcal{S}-\mathcal{T} & \mathcal{S}-\mathcal{U} \\ 0 & \mathcal{U}-\mathcal{T} & \mathcal{U}-\mathcal{S} & \mathcal{U}-\mathcal{S} & \mathcal{U}-\mathcal{T} & 0 \\ \mathcal{T}-\mathcal{U} & 0 & \mathcal{T}-\mathcal{S} & \mathcal{T}-\mathcal{S} & 0 & \mathcal{T}-\mathcal{U} \end{pmatrix}, \quad \delta = \begin{pmatrix} 2\mathcal{S} & 0 & 0 \\ 0 & 2\mathcal{U} & 0 \\ 0 & 0 & 2\mathcal{T} \end{pmatrix} \tag{B.4}$$

with

$$\mathcal{S} = \log\left(-\frac{s}{Q^2}\right), \qquad \mathcal{T} = \log\left(-\frac{t}{Q^2}\right), \qquad \mathcal{U} = \log\left(-\frac{u}{Q^2}\right). \tag{B.5}$$

We use eqs. (B.1) and (B.4) to show

$$\gamma|A^{(0)}\rangle = \left(-\frac{4iK}{stu}\right)2(sY - tX)\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \text{and} \quad \gamma\beta\begin{pmatrix}1\\1\\1\end{pmatrix} = 2\left(X^2 + Y^2 + Z^2\right)\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{B.6}$$

where

$$X = \log\left(\frac{t}{u}\right), \qquad Y = \log\left(\frac{u}{s}\right), \qquad Z = \log\left(\frac{s}{t}\right). \tag{B.7}$$

For consideration of the Regge limit $s \gg -t$, with $t < 0$ held fixed, it is convenient to set the arbitrary factorization scale Q^2 equal to $-t$, in which case the elements of the one-loop anomalous dimension matrix (B.4) take the form

$$
\alpha = \begin{pmatrix} Z & 0 & 0 & 0 & 0 & 0 \\ 0 & Z-X & 0 & 0 & 0 & 0 \\ 0 & 0 & -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X & 0 & 0 \\ 0 & 0 & 0 & 0 & Z-X & 0 \\ 0 & 0 & 0 & 0 & 0 & Z \end{pmatrix}, \qquad \beta = \begin{pmatrix} X & 0 & -Y \\ -X & Z & 0 \\ 0 & -Z & Y \\ 0 & -Z & Y \\ -X & Z & 0 \\ X & 0 & -Y \end{pmatrix}
$$

$$
\gamma = \begin{pmatrix} -Y & Z & 0 & 0 & Z & -Y \\ 0 & -X & Y & Y & -X & 0 \\ X & 0 & -Z & -Z & 0 & X \end{pmatrix}, \qquad \delta = \begin{pmatrix} 2Z & 0 & 0 \\ 0 & -2X & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.8}
$$

Finally, we analytically continue the variables X, Y, and Z to the physical region $s > 0$, $u,t < 0$, and then take $s \gg -t$ to obtain

$$
\begin{aligned}
X &\to -\log(-s/t) + \mathcal{O}(t/s) \\
Y &\to i\pi + \mathcal{O}(t/s) \\
Z &\to \log(-s/t) - i\pi + \mathcal{O}(t/s)
\end{aligned} \tag{B.9}
$$

From this, we see that the leading log behavior of the matrices (B.8) in the limit $s \gg -t$ is

$$
\alpha \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \log(-s/t), \qquad \beta \to \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \log(-s/t)
$$

$$
\gamma \to \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix} \log(-s/t), \qquad \delta \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \log(-s/t) \tag{B.10}
$$

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